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Uzawa-SOR Method for Fuzzy Linear System

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(Abstract) An Uzawa-SOR method is presented for solving fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is arbitrary fuzzy number vector. The convergence is analyzed and numerical example is given to illustrate the procedure.

Keywords: Iterative Method; Fuzzy Linear System; Uzawa-SOR

1. INTRODUCTION

Fuzzy linear system (FLS) has many applications in control problems, information, statistics, engineering, economics, finance and even social sciences. In the 1990s, Buckley et al. [8–10] investigated them in series. Rao and Chen [16] consider the numerical solution of FLSs in engineering analysis. Friedman et al. [12] consider a FLS as follows,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n, \end{cases} \quad (1.1)$$

where the coefficient matrix $A = (a_{ij})$ is a crisp matrix and $y = (y_i)$ is a fuzzy vector, $1 \le i, j \le n$. They suggest a model to solve this kind of fuzzy system. Based on it, numbers of numerical methods [1–7,11,13–15,17–19] have been presented for FLS (1.1). In this paper, we provide an iterative method named Uzawa-SOR (cf. [20]) for solving FLS (1.1).

The paper is organized as follows. In Section 2,we give some preliminaries for FLS (1.1). In Section 3, we propose the Uzawa-SOR method with the convergence theorem. An illustrative example is provided in Section 4 and the conclusion is drawn in Section 5.

2. PRELIMINARIES

Following [12], a fuzzy number is defined as $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, which satisfies,

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0, 1],
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0, 1],
- $\underline{u}(r) \le \overline{u}(r)),\ 0 \le r \le 1$.

To define a solution to (1.1) we should recall the arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$, $0 \le r \le 1$, and real number k,

(1) $x = y \Leftrightarrow \underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(3) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \ge 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.1. A fuzzy number vector $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)),\ \ 1 \le i \le n,\ 0 \le r \le 1,$$

is called a solution of FLS (1.1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i. \end{cases} \quad (2.1)$$

By (2.1), Friedman et al. [12] extend FLS (1.1) to a $2n \times 2n$ crisp linear system

$$SX = Y \quad (2.2)$$

where $S = (s_{kl})$, s_{kl} are determined as follows

$$\begin{matrix} a_{ij} \ge 0 \Rightarrow s_{ij} = a_{ij}, & s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, & s_{i+n,j} = -a_{ij}, \end{matrix} \quad 1 \le i,\ j \le n,$$

and any s_{kl} which is not determined by the above items is zero, $1 \le k, l \le 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix},\ Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

From [12], we know that S has the following structure

$$\begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$$

where $S_1, S_2 \geq 0,\ A = S_1 - S_2$, and (2.2) can be rewritten as follows

$$\begin{cases} S_1 \underline{X} - S_2 \overline{X} = \underline{Y}, \\ S_1 \underline{X} - S_2 \overline{X} = \overline{Y}, \end{cases} \tag{2.3}$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix},\ \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix},\ \underline{Y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_n \end{bmatrix},\ \overline{Y} = \begin{bmatrix} -\overline{y}_1 \\ -\overline{y}_2 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

The following theorem indicates when FLS (1.1) has a unique solution.

Theorem 2.2 [12]. *The matrix S is nonsingular if and only if the matrices* $A = S_1 - S_2$ *and* $S_1 + S_2$ *are both nonsingular.*

Under the conditions of Theorem 2.2, the solution $X = S^{-1}Y$ of (1.1) is thus unique but may still not be an *appropriate fuzzy vector*. By Theorem 2 of [12], we know that S^{-1} has the same structure like S, i.e.

$$S^{-1} = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}.$$

The following result provides a sufficient condition for the unique solution to be a fuzzy vector.

Theorem 2.3 [12]. The unique solution X of (2.2) is a fuzzy vector for arbitrary fuzzy vector Y, *if* S^{-1} is nonnegative.

Restricting the discussion to triangular fuzzy numbers, i.e. $\underline{y}_i(r), \overline{y}_i(r)$ and consequently $\underline{x}_i(r), \overline{x}_i(r)$ are all linear functions of r, and having calculated X which solves (2.2), we can define the fuzzy solution to the original system given by (1.1) as follows.

Definition 2.4. Let $X = \{(\underline{x}_i(r), -\overline{x}_i(r)), 1 \leq i \leq n\}$ denote the unique solution of (2.2). The fuzzy number vector $U = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n\}$ defined by

$$\underline{u}_i(r) = \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\},$$
$$\overline{u}_i(r) = \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}$$

is called the *fuzzy solution* of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r))$, $1 \leq i \leq n$ are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r)$, $\overline{u}_i(r) = \overline{x}_i(r), 1 \leq i \leq n$ and U is called a *strong fuzzy solution*, otherwise, U is called a *weak fuzzy solution*.

3. THE UZAWA-SOR METHOD

For the case S is nonsingular, without loss of generality, assume that $s_{ii} > 0, i = 1,2,\cdots,n,$ and $S_1 = D_1 - L_1 - U_1$, where $D_1 = \mathrm{diag}(s_{ii}), i = 1,2,\cdots,n,$ L_1 and U_1 are strictly lower and upper triangular matrices. Let $0 < \omega < 2$ be a relaxation parameter. Then for the first equation of (2.3), we take the following SOR iterative form:

$$(D_1 - \omega L_1)\underline{X}_{k+1} = [(1-\omega)D_1 + \omega U_1]\underline{X}_k + \omega(S_2 \overline{X}_k + \underline{Y}), \tag{3.1}$$

and for the second equation of (2.3), we can take the iterative scheme as follows,

$$\overline{X}_{k+1} = \overline{X}_k + \tau(S_2 \underline{X}_{k+1} - S_1 \overline{X}_k - \overline{Y}), \tag{3.2}$$

where τ is a real parameter.

Then we get the Uzawa-SOR method for FLS (1.1) as the following algorithm.

Algorithm 3.1. (Uzawa-SOR method) *Given initial vectors* $\underline{X}_0, \overline{X}_0 \in \mathrm{R}^n$, *a relaxation factor* $0 < \omega < 2$ *and a real parameter* τ. *For* $k = 0, 1, 2, \cdots,$ *the following iterative scheme is taken,*

$$\begin{aligned} \underline{X}_{k+1} &= (D_1 - \omega L_1)^{-1}[(1-\omega)D_1 + \omega U_1]\underline{X}_k \\ &\quad + \omega(D_1 - \omega L_1)^{-1}(S_2 \overline{X}_k + \underline{Y}), \\ \overline{X}_{k+1} &= \overline{X}_k + \tau(S_2 \underline{X}_{k+1} - S_1 \overline{X}_k - \overline{Y}). \end{aligned} \tag{3.3}$$

We have the following convergence theorem.

Theorem 3.2. *If* S_1 is symmetric positive definite, let λ_m and λ_M denote the smallest and the largest eigenvalues of S_1, then for $0 < \omega < 2$ and $0 < \tau < \frac{2}{\lambda_M}$, the Uzawa-SOR method (3.3) is convergent. The optimal parameter τ is $\tau_{\text{opt}} = \arg\min_{0<\tau<\frac{2}{\lambda_M}} \{|1 - \tau\lambda_M|, |1 - \tau\lambda_m|\}$.

Proof. Because S_1 is symmetric positive definite, for $0 < \omega < 2$, the SOR iteration (3.1) is convergent. For iteration (3.2), we can rewrite it as

$$\overline{X}_{k+1} = (I - \tau S_1)\overline{X}_k + \tau(S_2 \underline{X}_{k+1} - \overline{Y}).$$

The iteration matrix of (3.2) is $I - \tau S_1$. Suppose λ be an arbitrary eigenvalue of $I - \tau S_1$ and z be the corresponding eigenvector, we have

$$(I - \tau S_1)z = \lambda z,$$

that is

$$S_1 z = \frac{1-\lambda}{\tau} z.$$

Let λ_{S_1} be an arbitrary eigenvalue of S_1, thus $\lambda = 1 - \tau\lambda_{S_1}$. Therefore

$$|\lambda| \Leftrightarrow |1-\tau\lambda_{S_1}|<1 \Leftrightarrow \begin{cases} 1-\tau\lambda_{S_1}<1 \\ 1-\tau\lambda_{S_1}>-1 \end{cases}$$

$$\Leftrightarrow 0<\tau<\frac{2}{\lambda_{S_1}}.$$

Then we see that the iterative scheme (3.2) is convergent if $0<\tau<\frac{2}{\lambda_M}$, and the optimal τ is $\arg\min_{0<\tau<\frac{2}{\lambda_M}}\{|\lambda|\}$, i.e. $\arg\min_{0<\tau<\frac{2}{\lambda_M}}\{|1-\tau\lambda_M|,|1-\tau\lambda_m|\}$. □

4. NUMERICAL EXAMPLE

Example. Consider 2×2 fuzzy linear system

$$\begin{cases} x_1 - x_2 = (r, 2-r), \\ x_1 + 3x_2 = (4+r, 7-2r). \end{cases}$$

The exact solution is

$$\begin{cases} x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (1.375+0.625\,r, 2.875-0.875\,r), \\ x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (0.875+0.125\,r, 1.375-0.375\,r). \end{cases}$$

Direct calculations yield $\lambda_m = 0.8453$, $\lambda_M = 3.1547$, $\tau_{\text{opt}} = 0.5$.

With $\omega = 1, \tau = \tau_{\text{opt}} = 0.5$ and $X_0 = [0,0,0,0]^{\mathrm{T}}$, after 17 iterations, the numerical solution is

$$X_{1,0.5,17} = \begin{bmatrix} 1.3748+0.6251r \\ 0.8751+0.1250r \\ 2.8748-0.8749r \\ 1.3753-0.3751r \end{bmatrix},$$

i.e.

$$\begin{cases} x_1 = (1.3748+0.6251r, 2.8748-0.8749r), \\ x_2 = (0.8751+0.1250r, 1.3753-0.3751r). \end{cases}$$

With $\omega = 0.95, \tau = \tau_{\text{opt}} = 0.5$ and $X_0 = [0,0,0,0]^{\mathrm{T}}$, after 17 iterations, the numerical solution is

$$X_{0.95,0.5,17} = \begin{bmatrix} 1.3750+0.6250r \\ 0.8750+0.1250r \\ 2.8748-0.8749r \\ 1.3753-0.3751r \end{bmatrix},$$

i.e.

$$\begin{cases} x_1 = (1.3750+0.6250r, 2.8748-0.8749r), \\ x_2 = (0.8750+0.1250r, 1.3753-0.3751r). \end{cases}$$

By taking various ω and τ, we can get higher accuracy or faster convergence rate.

5. CONCLUSION

We present an Uzawa-SOR iterative method for $n\times n$ fuzzy linear system. If the proposed matrix *S* by Friedman et al. [12] is nonsingular, then for any initial vector X_0, the Uzawa-SOR iteration will converge to the unique solution of $SX = Y$. The numerical example shows that the method is effective, and with appropriate parameters ω and τ, it is expected to achieve higher accuracy or faster convergence rate than the known methods which have one or no parameter. As analyzed in Section 3, it is just obtained the convergence range for the symmetric positive definite case and the optimal parameter τ. The convergence for general case and the optimal ω need to be further investigated.

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